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Convergence of a class of discrete-time semiflows with applications to difference systems☆

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Abstract

In this paper, by employing comparison technique and invariance properties of a positively limited set, we investigate the convergence of precompact orbits of a class of discrete-time semiflows. In particular, we consider the convergence of precompact orbits of discrete-time semiflows generated by some monotone mapping. We then apply these abstract results to a class of difference systems to obtain the large-time behavior of solutions. Our results improve and extend some existing ones.

MSC: 34C12; 39A11

Keywords: Convergence; Difference system; Discrete-time semiflow; Positively limited set; Precompact orbit

1. Introduction

Recently, much progress has been made in applying the theory of monotone dynamical systems to investigate the problem of globally asymptotic behavior of continuous- and discrete-time semiflows. It is commonly hoped that most of the precompact orbits of a strongly monotone semiflow are convergent to a set of equilibria. For strongly monotone continuous-time semiflows, Hirsch [1,2] achieved this goal by

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employing the dichotomy of a positively limited set. For strongly monotone discrete-time semiflows, the limit set dichotomy theorem fails. We refer the reader to [3] for counter-examples about this. However, by making use of some additional hypotheses, convergence to an equilibrium for every precompact orbit of strongly monotone discrete-time semiflows is proved in the papers of [3–7]. Therefore, under certain hypotheses the dynamics of strongly monotone discrete-time semiflows are simple. It is natural to ask whether a similar conclusion holds for not strongly monotone or even non-monotone mappings. In this note we provide a positive answer to this question.

It should be mentioned that Wu [8] gave sufficient conditions for the convergence of the precompact orbits of a class of non-monotone discrete-time semiflows. But unfortunately, there are too many restrictions on the mapping which generates discrete-time semiflows. In fact, most of the restrictive conditions in [8] can be weakened or dropped (see Section 2 below for details). Huang and Yu [9] also investigated the problem of convergence of bounded orbits of a class of difference systems. For related results, we refer to [10–12]. However, it is not difficult to see that the abstract results of the paper [8] cannot be applied to the difference systems considered in [9].

Motivated by the results mentioned above, we study the asymptotic behavior of discrete-time semiflows generated by a class of non-monotone mappings in this paper. By comparing the positively limited set with some quasi-equilibrium (see Section 2 for more details on this definition) and applying invariance properties of a positively limited set, we obtain some results which improve and generalize the corresponding ones due to [8]. Our results show that some of the restrictive conditions in [8] can be weakened or dropped. Moreover, we also study the asymptotic behavior of precompact orbit of a class of monotone discrete-time semiflows. The obtained results improve the corresponding ones in [4].

In addition, we consider the applications of our abstract results to the following difference equation

$$x_n - x_{n-1} = -F(x_n) + G(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}), \qquad n \geq 1, \tag{1.1}$$

where k is a positive integer, $F : R^1 \longrightarrow R^1$ (throughout this paper, R^1 denotes the set of all real numbers) is continuous and nondecreasing, and $G : R^k \longrightarrow R^1$ is continuous. By transforming (1.1) equivalently into a class of discrete-time semiflows, we successfully give sufficient conditions for bounded orbits of (1.1) to converge to a constant. We also give the properties of unbounded orbits of (1.1). Obviously, (1.1) contains the following difference equation

$$x_n - x_{n-1} = -f(x_n) + g(x_{n-k}), \tag{1.2}$$

as a special case, where k is a positive integer, $f, g : R^1 \longrightarrow R^1$ are continuous, and f is nondecreasing on R^1. Asymptotic behavior of solutions of (1.2) with f strictly increasing on R^1 has been studied in [9]. In the present paper, as an application of our results, we obtain the same results for (1.2), but we take a rather different point of view in dealing with this problem and the obtained results also improve the corresponding ones in [9].

The paper is organized as follows. In Section 2, we develop some convergence results for a class of abstract discrete-time semiflows. In Section 3, we present some applications of these results obtained in Section 2.

2. Some general convergence results

Let X be a metrizable topological space endowed with a closed partial order relation $R \subseteq X \times X$ such that $\operatorname{Int} R \neq \phi$. For any $x, y, q \in X$ and any subset $A \subseteq X$, the following notations will be used: $x \leq y$ iff $(x, y) \in R$, $x < y$ if $(x, y) \in R$ and $x \neq y$, $x \ll y$ iff $(x, y) \in \operatorname{Int} R$, $A \leq q$ $(A < q)$ iff

$a \le q$ ($a < q$) for all $a \in A$, $q \le A$ ($q < A$) iff $q \le a$ ($q < a$) for all $a \in A$, $A \ll q$ iff $a \ll q$ for all $a \in A$, $q \ll A$ iff $q \ll a$ for all $a \in A$. We denote $\overline{Y}$ closure of set $Y \subseteq X$.

Consider a continuous mapping $S : X \longrightarrow X$. We define $E_S = \{e \in X : S(e) = e\}$. If $x \in X$, we define $O_S(x) = \{S^n(x) : n \ge 0\}$. If $\overline{O_S(x)}$ is compact, we define

$$\omega_S(x) = \bigcap_{j \ge 1} \overline{O_S(S^j(x))}.$$

One can observe that $\omega_S(x)$ is nonempty, compact, and invariant. In particular, the invariance of $\omega_S(x)$ will play a crucial role in the proofs of the main results of this paper. We will always assume that $I : R^1 \longrightarrow X$ is continuous, and $\tau_1 < \tau_2$ implies $I(\tau_1) \ll I(\tau_2)$. Throughout this section E denotes a closed subset of X together with $E_S \subseteq E$ and $I(R^1) \subseteq E$. For any given $e \in E$, we define $S^e = \{x \in X : x \ge e\}$ and $S_e = \{x \in X : x \le e\}$.

For convenience, we introduce the following assumptions.

(C1) For any $e \in E$, there exists an integer $N \ge 1$ such that $e \ll S^n(x)$ or $e = S^n(x)$ for any $x \in S^e$ and $n > N$.
(C2) For any $x \in X$, there exist $\alpha, \beta \in R^1$ such that $I(\alpha) \le x \le I(\beta)$.

If E satisfies assumption (C1), we call E the set of quasi-equilibria and call the point in E a quasi-equilibrium.

We are now in a position to state one of the main results of this section.

Theorem 2.1. *Let the mapping $S : X \longrightarrow X$ be continuous, S satisfy* (C1), *and assumption* (C2) *be satisfied. If $x \in X$ is given such that $\overline{O_S(x)}$ is compact, then $\omega_S(x) = \{I(\alpha^*)\}$ for some $\alpha^* \in R^1$.*

Proof. Since $\overline{O_S(x)}$ is compact, $\omega_S(x)$ is nonempty, compact, and invariant. Hence, by (C2), there exist $\alpha^*, \beta \in R^1$ such that $I(\alpha^*) \le \omega_S(x) \le I(\beta)$. Let $\alpha^* = \sup\{r \in R^1 : I(r) \le \omega_S(x)\}$. Then $\alpha^* \in R^1$. We want to show that $I(\alpha^*) \in \omega_S(x)$. Suppose not, i.e., $I(\alpha^*) < \omega_S(x)$. Then by assumption (C1) and the invariance of $\omega_S(x)$, we obtain $I(\alpha^*) \ll \omega_S(x)$. But this contradicts the definition of α^*. Therefore, $I(\alpha^*) \in \omega_S(x)$.

Next we will show that $\omega_S(x)^\sharp = 1$, where $\omega_S(x)^\sharp$ denotes the cardinal numbers of $\omega_S(x)$. Otherwise, $\omega_S(x)^\sharp > 1$. Then by assumption (C1) and the invariance of $\omega_S(x)$, there exist $q \in \omega_S(x)$ such that $q \gg I(\alpha^*)$. By the definition of $\omega_S(x)$, there exists $n_1 > 1$ such that $S^{n_1}(x) \gg I(\alpha^*)$. Because of the continuity of I, we can find a real number $\beta^* > \alpha^*$ such that $S^{n_1}(x) \gg I(\beta^*) \gg I(\alpha^*)$. By assumption (C1) and the fact that $I(\beta^*) \in E$, there exists $n_2 > n_1$ such that $S^n(x) \ge I(\beta^*) \gg I(\alpha^*)$ for all $n \ge n_2$. This implies that $\omega(x) \ge I(\beta^*) \gg I(\alpha^*)$, a contradiction to $I(\alpha^*) \in \omega_S(x)$. Therefore, $\omega_S(x) = \{I(\alpha^*)\}$. This completes the proof. $\square$

Remark 2.1. It is clear that Theorem 2.1 extends and improves Theorem 2.1 in [8] in many aspects such as: (a) We do not require that $I(R^1) \subseteq E_S$ holds; (b) assumption (ii) of Theorem 2.1 in [8] has been weakened by assumption (C1) drastically in our paper; (c) assumption (i) of Theorem 2.1 in [8] has been dropped in our Theorem 2.1.

Remark 2.2. It should be noted that the mapping I in assumption (C2) is not necessarily defined in R^1 itself. In fact, it may be defined in an arbitrary interval of R^1 such as [0, 1] and [0, 1).

In many applications, it is necessary to consider the symmetric form of Theorem 2.1. To do this, we introduce the following assumption.

(C3) For any $e \in E$, there exists an integer $N \geq 1$ such that $e \gg S^n(x)$ or $e = S^n(x)$ for any $x \in S_e$ and $n > N$.

Theorem 2.2. *Let the mapping $S : X \longrightarrow X$ be continuous, assumption* (C2) *be satisfied and S satisfy* (C3). *Then the conclusion of Theorem 2.1 holds.*

Proof. Let $\widetilde{R} = \{(x, y) \in X \times X : (y, x) \in R\}$. For any $\alpha \in R^1$, let $\widetilde{I}(\alpha) = I(-\alpha)$. Then replacing R and I in Theorem 2.1 by $\widetilde{R}$ and $\widetilde{I}$, respectively, we can conclude that S satisfies the conditions of Theorem 2.1, and so it follows from Theorem 2.1 that the conclusion of Theorem 2.2 holds. The proof is now complete. □

In [4], the convergence of discrete-time semiflows generated by strongly monotone mappings has been discussed (see Theorem 1.3 in [4] for more details). Before proceeding, we need the following assumption.

(C4) Let the mapping $T : X \longrightarrow X$ be continuous. There exists an integer $N \geq 1$ such that for any $x, y \in X$ with $x \geq y$, $T^n(x) \gg T^n(y)$ or $T^n(x) = T^n(y)$ for $n \geq N$.

We say that T is semi-strongly monotone if T satisfies assumption (C4).

As a direct consequence of Theorems 2.1 and 2.2, we obtain the following convergence of discrete-time semiflows generated by semi-strongly monotone mappings, which improves Theorem 1.3 in [4].

Corollary 2.1. *Let the mapping $T : X \longrightarrow X$ be continuous and semi-strongly monotone, also let assumption* (C2) *be satisfied and $I(R^1) \subseteq E_T$. If $x \in X$ is given such that $\overline{O_T(x)}$ is compact, then $\omega_T(x) = \{I(\alpha)\}$ for some $\alpha \in R^1$.*

Proof. Let $E = E_T$. It is not difficult to check that T satisfies the conditions of Theorem 2.1 or 2.2. Therefore, we can apply Theorem 2.1 or 2.2 to obtain the conclusion of Corollary 2.1. This completes the proof. □

3. Applications to some difference equations

In this section, we apply the abstract results in Section 2 to consider the large-time behavior of solutions for the difference equations (1.1) and (1.2).

To simplify the following argument, we introduce the following auxiliary mappings and establish several important lemmas that will play a major role in our analysis.

Let F and G be defined as in (1.1). We define the mappings

$$\varphi : R^1 \longrightarrow R^1 \text{ by } \varphi(x) = x + F(x)$$

and

$$H : R^k \longrightarrow R^k \text{ by } H(z_1, z_2, \ldots, z_k) = (z_2, \ldots, z_k, \varphi^{-1}(z_k + G(z_1, z_2, \ldots, z_k))). \tag{3.1}$$

It follows that $\varphi(x)$ and $\varphi^{-1}(x)$ are continuous and strictly increasing on R^1, and hence $H(z)$ is continuous on R^k.

Lemma 3.1. *Let the mapping φ be defined as above, α be a given constant and define the mapping $\gamma : R^1 \longrightarrow R^1$ by $\gamma(x) = \varphi^{-1}(x + F(\alpha))$. Then for any given $M > 0$, there exists $N > M$ such that $\gamma^i(N) > M$ for all $i \in \{1, 2, \ldots, k\}$.*

Proof. Let $p(x) \equiv \min_{1\leq i\leq k} \gamma^i(x)$, where $x \in R^1$. Since $\lim_{x\longrightarrow+\infty} \gamma(x) = +\infty$, it follows that $\lim_{x\longrightarrow+\infty} p(x) = +\infty$. Therefore, the conclusion of the lemma is true. $\square$

In what follows, we will use R^k_+ to denote the set of all nonnegative vectors in R^k. It then follows that R^k_+ is an order cone in R^k. For any $u, v \in R^k$, the following notations will be used: $u \leq v$ iff $v-u \in R^k_+$, $u < v$ iff $u \leq v$ and $u \neq v$, $u \ll v$ iff $v - u \in \text{Int } R^k_+$.

Lemma 3.2. *Let the mappings F, G and H be as above. If $G(z) \geq F(\alpha)$ for all $\alpha \in R^1$ and $z \in R^k$ such that $z \geq (\alpha, \alpha, \ldots, \alpha) \in R^k$, then we have*

(i) *if $z \in R^k$ is given such that $\overline{O_H(z)}$ is compact, then there exists $c \in R^1$ such that*

$$\lim_{n\longrightarrow\infty} H^n(z) = (c, c, \ldots, c) \in R^k,$$

where $H^n(z) = ((H^n(z))_1, (H^n(z))_2, \ldots, (H^n(z))_k) = H(H^{n-1}(z))$ for $n = 1, 2, \ldots,$ and $H^0 \equiv Id_{R^k}$, in which Id_{R^k} denotes the identical mapping from R^k to R^k;

(ii) *if $z \in R^k$ is given such that $O_H(z)$ is unbounded, then*

$$\lim_{n\longrightarrow\infty} (H^n(z))_i = +\infty \qquad \text{for all } i \in \{1, \ldots, k\}.$$

Proof. Let $E = \{(\alpha, \alpha, \ldots, \alpha) \in R^k : \alpha \in R^1\}$ and define the mapping $I : R^1 \longrightarrow R^k$ by $I(\alpha) = (\alpha, \alpha, \ldots, \alpha) \in R^k$. By the definition of E and I, assumption (C2) holds. Assume that $e = (\alpha, \alpha, \ldots, \alpha) \in E$, $z \in R^k$ and $z \geq e$. We want to show that

$$z_k > \alpha \text{ implies } (H(z))_k > \alpha. \tag{3.2}$$

Indeed, by the definition of H, we get

$$\begin{aligned}(H(z))_k &= \varphi^{-1}(z_k + G(z))\\ &\geq \varphi^{-1}(z_k + F(\alpha))\\ &> \varphi^{-1}(\alpha + F(\alpha))\\ &= \alpha.\end{aligned}$$

Hence, by (3.2) and the continuity of H, we have $H(z) \geq e$. We will prove that

$$H^n(z) = e \qquad \text{or} \qquad H^n(z) \gg e, \qquad \text{for all } n \geq 2k + 2. \tag{3.3}$$

We next distinguish two cases to finish the proof of (3.3).

Case 1. $H^{k+1}(z) = e$.

Let $n_0 = \inf\{n \geq k + 1 : H^n(z) > e\}$. If $n_0 = +\infty$, then $H^n(z) = e$ for all $n \geq k + 1$, and hence, the proof is complete. If $n_0 < +\infty$, then we can conclude $n_0 = k + 2$. Otherwise, $H^{k+2}(z) = e$ and $n_0 > k + 2$. Thus, $H^{k+2}(z) = H(H^{k+1}(z)) = H(e) = e$, and so $H^{n_0}(e) = e$, a contradiction. By the definition of n_0, we have $H^{k+2}(z) = H(H^{k+1}(z)) = H(e) > e$, and hence, $(H^{k+2}(z))_k > \alpha$. Therefore, from (3.2), we get $H^n(z) \gg e$, for all $n \geq 2k + 2$.

Case 2. $H^{k+1}(z) > e$.

It follows that there exists $i \in \{1, 2, \ldots, k\}$ such that $(H^{k+1}(z))_i > \alpha$. By the definition of H, we obtain $(H^{i+1}(z))_k > \alpha$. Hence, from (3.2), we get $H^n(z) \gg e$ for all $n \geq k + i$.

From the above discussion, we can conclude that (3.3) holds and hence, H satisfies assumption (C1). Therefore, the conclusion (i) is a consequence of Theorem 2.1.

We next show that the conclusion (ii) holds. Indeed, let $z \in R^k$ be given such that $O_H(z)$ is unbounded. Then choose $\beta \in R^1$ such that $z \geq (\beta, \beta, \ldots, \beta) \in R^k$, from which it follows that $H^n(z) \geq (\beta, \beta, \ldots, \beta) \in R^k$. By Lemma 3.1 and the fact that $O_H(z)$ is unbounded, for any $M > 0$, there exists $n_1 > 1$ such that

$$H^{n_1}(z) \geq (M, M, \ldots, M) \in R^k.$$

Hence,

$$H^n(z) \geq (M, M, \ldots, M) \in R^k \qquad \text{for all } n \geq n_1.$$

Therefore, $\lim_{n\longrightarrow\infty}(H^n(z))_i = +\infty$ for all $i \in \{1, \ldots, k\}$. This proves the lemma. □

Remark 3.1. It should be noted that Theorem 2.1 in [8] cannot be applied to the mapping H in Lemma 3.2.

Theorem 3.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). If $G(z) \geq F(\alpha)$ for all $z \in R^k$ and $\alpha \in R^1$ such that $z \geq (\alpha, \alpha, \ldots, \alpha) \in R^k$, then either $\lim_{n\longrightarrow\infty} x_n = +\infty$ or $\lim_{n\longrightarrow\infty} x_n = c$ for some $c \in R^1$.*

Proof. Note that system (1.1) is equivalent to the system

$$z^{(n)} = H^n(z),$$

where H is defined as (3.1). The desired conclusion follows immediately from Lemma 3.2 and thus, the proof is complete. □

We are now ready to state a symmetric form of Theorem 3.1, the proof of which is similar to that of Theorem 3.1 and therefore, it is omitted.

Theorem 3.2. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.1). If $G(z) \leq F(\alpha)$ for all $z \in R^k$ and $\alpha \in R^1$ such that $z \leq (\alpha, \alpha, \ldots, \alpha) \in R^k$, then either $\lim_{n\longrightarrow\infty} x_n = -\infty$ or $\lim_{n\longrightarrow\infty} x_n = c$ for some $c \in R^1$.*

As an application of Theorems 3.1 and 3.2, for the special case of (1.1), we can get the following results.

Corollary 3.1. *Let f and g be defined as in (1.2) and let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.2). If $g(x) \geq f(x)$ for all $x \in R^1$, then either $\lim_{n\longrightarrow\infty} x_n = +\infty$ or $\lim_{n\longrightarrow\infty} x_n = c$ for some $c \in R^1$.*

Proof. Let $F(x) = f(x)$ for any $x \in R^1$ and $G(z_1, z_2, \ldots, z_k) = g(z_1)$ for any $(z_1, z_2, \ldots, z_k) \in R^k$. Then, by exploiting Theorem 3.1, the conclusion of Corollary 3.1 is immediate. □

Corollary 3.2. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.2). If $f(x) \geq g(x)$ for all $x \in R^1$, then either $\lim_{n\longrightarrow\infty} x_n = -\infty$ or $\lim_{n\longrightarrow\infty} x_n = c$ for some $c \in R^1$.*

Proof. The proof is similar to that of Corollary 3.1, and so it is omitted. □

Corollary 3.3. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of (1.2). If $f \equiv g$, then $\lim_{n\longrightarrow\infty} x_n = c$ for some $c \in R^1$.*

Proof. Applying Corollaries 3.1 and 3.2, we can conclude that Corollary 3.3 holds. □

Remark 3.2. Corollary 2.1 can also be applied to conclude that Corollary 3.3 holds. Indeed, define the following auxiliary mapping $\psi : R^1 \longrightarrow R^1$ by $\psi(x) = x + f(x)$. It follows that ψ and ψ^{-1} are continuous and strictly increasing on R^1. Also, define the mapping

$$h : R^k \longrightarrow R^k \text{ by } h(z_1, z_2, \ldots, z_k) = (z_2, z_3, \ldots, z_k, \psi^{-1}(z_k + f(z_1))).$$

We claim that h satisfies assumption (C4). Indeed, assume that $z, z' \in R^k$ satisfy $z \geq z'$. Since ψ is strictly increasing and f is nondecreasing, it follows from the definition of h that $h(z) \geq h(z')$. We next distinguish two cases to finish the proof of the above claim.

Case 1. $h^{k+1}(z) = h^{k+1}(z')$.

It follows that $h^n(z) = h^n(z')$ for all $n \geq k+1$, and hence the above claim is established.

Case 2. $h^{k+1}(z) > h^{k+1}(z')$.

It follows that there exists $i \in \{1, 2, \ldots, k\}$ such that $(h^{k+1}(z))_i > (h^{k+1}(z'))_i$. Hence, by the definition of h, we have $(h^{i+1}(z))_k > (h^{i+1}(z'))_k$. Again by the definition of h, we get

$$\begin{aligned}(h^{i+2}(z))_k &= \psi^{-1}((h^{i+1}(z))_k + f((h^{i+1}(z))_1)) \\ &> \psi^{-1}((h^{i+1}(z'))_k + f((h^{i+1}(z'))_1)) \\ &= (h^{i+2}(z'))_k.\end{aligned}$$

Thus, by induction, we have $h^n(z) \gg h^n(z')$ for all $n \geq k+i$.

Therefore, from the above discussion, we can conclude that h satisfies assumption (C4). It then follows from Corollary 2.1 that Corollary 3.3 is established.

Remark 3.3. Corollaries 3.1–3.3 improve the results obtained in [9] since f is required to be strictly increasing in [9]. In particular, our proofs are quite different from those of [9]. We refer to [9] for a detailed description of the applications of Corollaries 3.1–3.3.

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